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Sturm–Liouville problems with boundary conditions depending quadratically on the eigenparameter

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Abstract

We study Sturm–Liouville problems with right-hand boundary conditions depending on the spectral parameter in a quadratic manner. A modified Crum–Darboux transformation is used to produce chains of problems almost isospectral with the given one. The problems in the chain have boundary conditions which in various cases are affine or bilinear in the spectral parameter, and in all cases culminate in a problem with constant boundary conditions. This extends recent work of Binding, Browne, Code and Watson when the right-hand condition is either an affine function of the spectral parameter with negative leading coefficient or a Herglotz function.

Keywords: Sturm–Liouville; Eigenparameter dependent boundary conditions

1. Introduction

In this article we shall study the Sturm–Liouville problem:

$$-y'' + qy = \lambda y, \tag{1}$$

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$$\frac{y'}{y}(0) = \cot\alpha, \tag{2}$$

$$\frac{y'}{y}(1) = a\lambda^2 + b\lambda + c. \tag{3}$$

Here q is an integrable function on [0, 1] and (2) is to be interpreted as the Dirichlet condition $y(1) = 0$ should $\alpha = 0$. The coefficients a, b, c are real and we shall allow a to be either positive or negative. In this sense this paper is both a companion and a sequel to [1] where (3) was replaced by the affine function $R(\lambda) = a\lambda + b$, $a < 0$. Interest in Sturm–Liouville problems with boundary conditions depending on the eigenparameter has a considerable history which has intensified in recent years with particular attention being paid to cases in which the boundary condition at $x = 1$ involves various types of functions—see, e.g., [2,4,5,7] where references to earlier work are provided. Thus far, most attention has been devoted to circumstances in which the function $R(\lambda)$ used in the boundary condition (3) is of Herglotz type

$$R(\lambda) = a\lambda + b + \sum_{j=1}^{N} \frac{b_j}{c_j - \lambda}, \quad a \geqslant 0,\ b_j > 0,\ j = 1, \dots, N.$$

While R may have a finite number of poles and thus (for real λ) have a graph consisting of a number of branches, it is increasing on each of these branches. This makes it particularly suited for analysis via Prüfer angle techniques which lead quickly to results concerning eigenvalue asymptotics and oscillation theory of eigenfunctions. Much has been made of this in the above citations.

In a recent series of papers Binding, Browne and Watson [2,4–6] have studied the use of the Crum–Darboux transformation on these problems. This transformation, developed by Crum [10] but dating to Darboux, acts on a given Sturm–Liouville problem of the above type with a Herglotz function R and produces a new problem of similar type with a simpler function R where either a is replaced by 0 or N is reduced by 1. An important feature is that the new problem is isospectral with the original (or almost so in that the two problems have spectra which coincide except for finitely many eigenvalues). The transformation, or repeated applications of it, reduces the problem to an almost isospectral one having classical boundary conditions (i.e., conditions for which R is constant) at $x = 1$. This work, while interesting and significant in its own right in that it produces important new classes of isospectral Sturm–Liouville problems of classical and eigenvalue-dependent types, has also provided the necessary background for consideration of inverse spectral problems with such eigenparameter boundary conditions. In this sense, the transformation has served two fundamental purposes and interest in it and its uses continues.

Focus has naturally turned to cases in which the increasing nature of the function R is not demanded; the situation is now much more complicated, with the possibility of non-real eigenvalues and of eigenvalues with algebraic multiplicity exceeding 1 being present. The first contribution to this new area, and the companion to this paper, is to be found in [1], where the thrust is to investigate the use of the Crum–Darboux transformation in the case when $R(\lambda) = a\lambda + b$, $a < 0$. While the expected complications do arise, the interesting point is that the Crum–Darboux transformation can, with suitable but non-trivial modifications, again be used to produce (almost) isospectral problems with constant boundary conditions thus once more providing new classes of (almost) isospectral problems and also setting the stage for inverse spectral theory.

Here we shall take R to be quadratic in λ as in (3). Our aim will be to demonstrate that the (modified) Crum–Darboux transformation can again be applied in this situation, producing, after suitably many iterations, problems with constant boundary conditions. As with the previous studies, we find this interesting (and mathematically delicate because of the large number of possibilities which can arise) in its own right. Of course the description and classification of Sturm–Liouville problems which are isospectral (or almost isospectral) is one of significance and long standing interest. Here we shall have a chain of such problems commencing with one having a quadratic type condition at $x = 1$ as above, linked to another whose condition at $x = 1$ usually involves a bilinear function, then to another with an affine function of λ, and finally to a problem with constant boundary conditions. Another interesting aspect is that the quadratic function in (3) has a region where its graph is decreasing and a region where it is increasing, so that in some sense it is a hybrid type of situation. To our knowledge, this is the first such example to be studied in depth. Nonetheless, the Crum–Darboux transformation can still be employed to advantage. Moreover, we see this a first step towards the study of inverse spectral problems for situations in which R is a quadratic. We propose to take up that topic in a later article.

The plan of the paper is as follows. In Section 2, we briefly describe the background results giving existence and asymptotics of the eigenvalues of (1)–(3) as well as the Prüfer angle theory required to approach the problem. We list results from [1] for completeness here, but refer the reader to that reference for details. We list the various possibilities (24 in all) which can arise in terms of the existence of non-real eigenvalues and of eigenvalues of algebraic multiplicity exceeding 1. A systematic coding of these possibilities is given. Section 3 reviews the definition of, and results on, the Crum–Darboux transformation. The later sections apply the transformation in the various cases and describe the manner in which the so-called quadratic boundary value problem (1)–(3) can be reduced to an almost isospectral problem with constant boundary conditions, so achieving our aim. Much of the material in this paper is based on the MSc thesis of the first author [3].

2. Preliminaries

The main result concerning the existence and asymptotics of eigenvalues for our problem is taken from [2]:

Theorem 1. *The problem* (1)–(3) *has a sequence of eigenvalues* λ_n, $n \geqslant 0$, *which, when listed according to increasing real part and repeated according to algebraic multiplicity, satisfy*

(i) *for* n *sufficiently large, all eigenvalues are real and algebraically simple,*
(ii) *there is at most one conjugate complex pair of non-real eigenvalues,*
(iii) $$\lambda_n = \begin{cases} (n-\frac{3}{2})^2\pi^2 + 2\cot\alpha + \int_0^1 q + o\left(\frac{1}{n}\right), & \alpha \neq 0, \\ (n-1)^2\pi^2 + \int_0^1 q + o\left(\frac{1}{n}\right), & \alpha = 0, \end{cases}$$

(iv) $\lambda_n \sim \lambda_{n-2}^D$ *where* λ_n^D, $n \geqslant 0$ *are the eigenvalues to the problem* $(1,2)$ *with the Dirichlet condition* $y(1)=0$. *When* $a>0$ (*respectively* $a<0$), $\lambda_n > \lambda_{n-2}^D$ (*respectively* $\lambda_n < \lambda_{n-2}^D$) *for n sufficiently large.*

Prüfer angle theory for this equation rests on the first-order non-linear initial value problem

$$\theta'(\lambda,x) = \cos^2\theta(\lambda,x) + \big(\lambda - q(x)\big)\sin^2\theta(\lambda,x), \qquad \theta(\lambda,0)=\alpha,$$

the connection with (1) being via $\cot\theta = y'/y$. This approach to eigenvalue theory is by now quite customary, but a suitable reference for results in this area is [9]. It suffices here to point out that the so-called "Prüfer curve" given as a function of the (real) variable λ by $\cot\theta(\lambda,1)$ has poles at $\lambda=\lambda_n^D$, $n\geqslant 0$, and is decreasing on each of its branches. Real eigenvalues for (1)–(3) correspond to intersections between this curve and the quadratic $R(\lambda)$. Moreover, eigenvalues of multiplicity $k>1$ correspond to λ-values where the quadratic and the Prüfer curve have contact of order $k-1$. Details (for the case of affine R) are given in [1], and [8] provides additional material. The branches of the Prüfer curve, which we label B_n, $n\geqslant 0$, correspond to λ-intervals $(-\infty,\lambda_0^D]$, $(\lambda_0^D,\lambda_1^D],\ldots.$ From this short discussion, we see that a number of possibilities can arise regarding the intersections of the Prüfer curve and the quadratic $R(\lambda)$. This also enables us to provide a systematic coding of the various types of situations which can arise, extending and complementing the coding for affine problems given in [1].

The first element of the code will consist of either D or N to indicate $\alpha=0$ (a Dirichlet condition) or $\alpha\neq 0$ (a non-Dirichlet condition) at $x=0$, respectively. Secondly, we shall use Q^+,Q^-, to distinguish between $a>0$, $a<0$ respectively. The notations $\mathrm{A}^\pm$, $\mathrm{B}^\pm$ will also be used for problems with an affine (respectively bilinear) condition at $x=1$: $R(\lambda)=a\lambda+b$ (respectively $R(\lambda)=(a\lambda+b)/(c\lambda+d)$), the distinction being made again between $a>0$, $a<0$ (respectively between $\delta>0$, $\delta<0$ where $\delta=ad-bc$). The third element of the code will describe the nature of intersections between the Prüfer curve and the quadratic R.

When $a>0$, the opening branch, B_0, of the Prüfer curve will have either two intersections counted according to multiplicity, or none with the quadratic. The case of two simple intersections is coded as $(1,1)_0$ and that of a double intersection as $(2)_0$. When there are no intersections on B_0, it is possible for a later branch B_k, $k>0$, to have three intersections counted according to multiplicity. These can occur as three simple intersections—coded as $(1,1,1)_k$, as a simple intersection followed by a double intersection—coded $(1,2)_k$, as a double intersection followed by a simple intersection—coded $(2,1)_k$, or as a triple intersection—coded $(3)_k$. The remaining possibility when B_0 has no intersections, is for each B_k, $k>0$, to have one simple intersection and in this case the problem (1)–(3) has a conjugate pair of non-real eigenvalues. We code this as $\mathbb{C}$. As an example, $\mathrm{DQ}^+(1,2)_k$ would code a problem for which $\alpha=0$, the quadratic R has $a>0$, there are no intersections on B_0, and on some branch B_k, $k>0$, there are three intersections which, in accord with our listing of eigenvalues, would correspond to $\lambda_{k-1}<\lambda_k=\lambda_{k+1}$. For $a>0$, we have a total of 14 different cases.

When $a<0$, we note first that there will always be one intersection on B_0. It is possible for any branch B_k, $k\geqslant 0$, to have three intersections counted according to multiplicity, and we use the codings as above with $k \geqslant 0$. Alternatively, there may be one simple intersection on each branch and a conjugate pair of non-real eigenvalues, which we code again as $\mathbb{C}$. This gives rise to 10 individual cases. As an example $\mathrm{NQ}^-\mathbb{C}$ would describe a problem with $\alpha > 0, a < 0$ having a pair of non-real and a sequence of real eigenvalues. Other examples are easy to construct and interpret. The net result is that there are 24 different possibilities for the problem (1)–(3).

3. The Crum–Darboux transformation

Much of the material here can be found in [1], but we list it here for completeness and ease of reference.

Definition 2. Suppose the functions $f_1, \ldots, f_k$ satisfy (1) with $\lambda = \lambda_1, \ldots, \lambda_k$, respectively. The Wronskian $W(f_1, \ldots, f_k)$ is defined as the $k \times k$ determinant

$$W(f_1, \ldots, f_k)(x) = \det\left| f_j^{(i-1)}(x)\right|_{i,j=1,\ldots,k},$$

where the differential equation (1) is used to replace second- and higher-order derivatives of the f_j in terms of f_j, f_j', and λ_j for $j = 1, \ldots, k$.

When $W(f_1, \ldots, f_k)(x) \neq 0$ on $[0, 1]$ we define the Crum–Darboux transformation with base functions $f_1, \ldots, f_k$ as the mapping which, for a given function y satisfying (1), produces the function e where

$$e(x) = \frac{W(f_1, \ldots, f_k, y)(x)}{W(f_1, \ldots, f_k)(x)}. \tag{4}$$

The following important properties of this transformation are detailed in [1].

Theorem 3. *Suppose that $f_1, \ldots, f_k, y$ are solutions of* (1) *with distinct simple eigenparameter values $\mu_1, \ldots, \mu_k, \lambda$, respectively. Then with e defined as in* (4),

(i) $$\frac{e'}{e}(x) = \frac{W'(f_1, \ldots, f_k, y)(x)}{W(f_1, \ldots, f_k, y)(x)} - \frac{W'(f_1, \ldots, f_k)(x)}{W(f_1, \ldots, f_k)(x)},$$

(ii) $$-e'' + \left(q - 2\left(\frac{W'(f_1, \ldots, f_k)}{W(f_1, \ldots, f_k)}\right)'\right)e = \lambda e,$$

wherever $W(f_1, \ldots, f_k)(x) \neq 0$,

(iii) *if $k \geqslant 2$, with*

$$\psi = \frac{W(f_1, \ldots, f_{k-1})}{W(f_1, \ldots, f_k)},$$

we have

$$-\psi'' + \left(q - 2\left(\frac{W'(f_1, \ldots, f_k)}{W(f_1, \ldots, f_k)}\right)'\right)\psi = \mu_k \psi,$$

wherever $W(f_1, \ldots, f_k)(x) \neq 0$.

Theorem 4. *Suppose $\hat{\lambda}$ is an eigenvalue of* (1)–(3) *with algebraic multiplicity $m > 1$ and with eigenfunction f_0 and associated eigenfunctions $f_1, \dots, f_{m-1}$ and suppose also that z is a function satisfying* (1) *with $\lambda = \mu$. Then the system of functions $w_0, \dots, w_{m-2}$ defined where $W(z, f_0)(x) \neq 0$ by*

$$w_{j-1} = \frac{W(z, f_0, f_j)}{W(z, f_0)}, \quad j = 1, \dots, m-1,$$

satisfies

$$L(w_0) = 0,$$
$$L(w_j) = w_{j-1}, \quad j = 1, \dots, m-2,$$

where

$$L(y) = -y'' + \left(q - 2\left(\frac{W'(z, f_0)}{W(z, f_0)}\right)' - \hat{\lambda}\right)y.$$

Note that the result remains valid if z is omitted or replaced by $z_1, \dots, z_n$ corresponding to $\mu_1, \dots, \mu_n$ for some $n \geqslant 1$.

We are now ready to apply the Crum–Darboux transformation to the various possibilities arising from the problem (1)–(3).

4. Transformations with one base function

This section covers the simplest cases of transformation to a classical Sturm–Liouville problem: viz. $NQ^+(1,1)_0$, $NQ^+(2)_0$ and $DQ^+(1,1)_0$. In these cases, the spectrum of (1)–(3) consists of a real sequence $\lambda_0 \leqslant \lambda_1 < \lambda_2 < \cdots$ with $\lambda_0 = \lambda_1$ occurring only in case $NQ^+(2)_0$. We take the corresponding eigenfunctions to be y_n, $n \geqslant 0$ with y_1 being interpreted as the associated eigenfunction for the double eigenvalue $\lambda_0 = \lambda_1$ in $NQ^+(2)_0$.

For $NQ^+(1,1)_0$ and $NQ^+(2)_0$ we take $f_1 = y_0$ as the single base function for a Crum–Darboux transformation and define

$$e_n = \frac{W(f_1, y_n)}{f_1}, \quad n \geqslant 1. \tag{5}$$

It is important to note that y_0 has no zeros in $[0,1]$ so that the e_n are well-defined.

For $DQ^+(1,1)_0$ we see that y_0 vanishes at $x = 0$, so that a modification to the above technique is required. We employ a perturbation approach which will be used frequently in later sections. Prüfer theory shows that $\cot\theta(\lambda, 1)$ is decreasing in the initial condition α and so we select $\alpha_1 > 0$ sufficiently small so that the Prüfer curve $\cot\theta(\lambda, 1)$ with initial condition α_1 intersects the parabola $R(\lambda)$ for two distinct values of λ lying between λ_0 and λ_1. We take μ as the smaller of these values and take f_1 to satisfy

$$-f_1'' + qf_1 = \mu f_1, \qquad \frac{f_1'}{f_1}(0) = \cot\alpha_1, \qquad \frac{f_1'}{f_1}(1) = R(\mu).$$

Note that f_1 has no zeros in $[0,1]$. This function f_1 will be the single base function in this case and we define e_n as in (5) above but with $n \geqslant 0$.

Theorem 5.

(i) *For* $\mathrm{NQ}^+(1,1)_0$ *and* $\mathrm{NQ}^+(2)_0$, *the transformation* (5) *with* $n \geqslant 1$ *generates the eigenfunctions to a Sturm–Liouville problem with potential*

$$Q = q - 2\left(\frac{f_1'}{f_1}\right)'$$

and boundary conditions

$$y(0) = 0, \qquad \frac{y'}{y}(1) = \frac{-a\gamma\lambda - (1 + \gamma(a\lambda_0 + b))}{a\lambda + a\lambda_0 + b}, \tag{6}$$

where $\gamma = R(\lambda_0)$. *The transformed problem has real simple spectrum* λ_n, $n \geqslant 1$.

(ii) *For* $\mathrm{DQ}^+(1,1)_0$ *the transformed problem has potential* Q *as above and boundary conditions* $\frac{y'}{y}(0) = \cot\alpha_1$, *and* (6) *with* $\gamma = R(\mu)$. *The spectrum is given by* λ_n, $n \geqslant 0$.

In all cases, the bilinear form giving the transformed boundary condition at $x = 1$ *has discriminant* $a > 0$.

Proof. Straightforward calculations using the results of the previous sections verify the claims that the e_n satisfy the transformed Sturm–Liouville equations and the boundary conditions are routinely verified (although some care must be taken for the $\mathrm{NQ}^+(2)_0$ case). In addition, the asymptotics for λ_n and the expected eigenvalue asymptotics for the transformed problem (see [2,7]) verify that the spectrum of the transformed problem is as stated. □

The upshot of this theorem is that the Crum–Darboux transformation, when applied as above to any of the three types $\mathrm{NQ}^+(1,1)_0$, $\mathrm{NQ}^+(2)_0$ or $\mathrm{DQ}^+(1,1)_0$, produces an isospectral or almost isospectral problem of type DB^+ for the first two possibilities or NB^+ for the third. These problems are of the Herglotz type discussed in Section 1 and as such we can appeal to the results of [4,5] and apply further Crum–Darboux transformations—each with a single base function—to come to an almost isospectral problem with constant boundary conditions. This chain of transformations will proceed via problems of the form NA^+, DA^+. The ultimate outcome is that these cases of (1)–(3) are linked directly in an almost isospectral manner to classical problems.

5. Transformations with two base functions

This section covers the majority of the cases described in Section 2. A key result from [1] is

Theorem 6. *Let* u, z *be solutions of* (1), (2) *with* λ *replaced by* μ *and* ξ, *and* α *replaced by* β *and* γ, *respectively. Suppose that* u *and* z *have the same number of zeros in* $(0,1)$. *If* $\beta > \gamma$ *and* $\mu > \xi$, *then* $W(u,z)$ *is non-zero everywhere on* $[0,1]$.

Throughout this section the spectrum will consist of real eigenvalues. Additionally, y_j will denote the eigenfunction associated with a simple eigenvalue λ_j, and in the case of a multiple eigenvalue $\lambda_j = \lambda_{j+1} = \cdots = \lambda_{j+p}$, we use $y_j, y_{j+1}, \ldots, y_{j+p}$ to denote the eigenfunction and associated eigenfunctions, respectively.

Firstly, we list a number of cases which can be approached by a perturbation technique similar to that of the previous section. In each case, we select for the first base function f_1 an eigenfunction of (1)–(3) corresponding to an eigenvalue λ_j and for f_2 an eigenfunction corresponding to an eigenvalue μ from a perturbed problem with a new initial condition α_1 obtained by a small change in α. As before, the new Prüfer curve using α_1 intersects $R(\lambda)$ at a point μ close to λ_j. The two eigenfunctions (f_1, f_2) will have the same number of zeros in $(0,1)$ and the relation between λ_j and μ, and between α and α_1 is chosen so that Theorem 6 applies. The method and the cases under consideration here are collected in Table 1 as follows. We indicate the case under consideration using our code, the nature of the spectrum as it applies to intersections of the kth branch B_k of the Prüfer curve and R, the choice of λ_j and hence of $y_j = f_1$, whether α_1 is obtained by an increase or decrease in α, and whether μ is greater than or less than λ_j. It is instructive and helpful for the reader to sketch typical situations as an aid to understanding the table.

The upshot of these remarks is that in each of the cases listed above, we can appeal to Theorem 6 and claim that $W(f_1, f_2)$ does not vanish on $[0,1]$ and hence we can form a Crum–Darboux transformation by setting

$$e_n = \frac{W(f_1, f_2, y_n)}{W(f_1, f_2)}, \quad n \geqslant 0,\ n \neq j \text{ as per the table.} \tag{7}$$

We can calculate the boundary conditions satisfied by the functions e_n with the help of Theorem 3(i) as:

$$\frac{e_n'}{e_n}(0) = \begin{cases} \cot\alpha - \frac{\mu-\lambda_j}{\cot\alpha-\cot\alpha_1}, & \text{if } \alpha \neq 0, \\ 0, & \text{if } \alpha = 0, \end{cases} \tag{8}$$

$$\frac{e_n'}{e_n}(1) = \gamma\lambda_n + \frac{b\gamma}{a} - a\lambda_j + c - \frac{1}{\gamma}, \tag{9}$$

where $\gamma = -(a\lambda_j + a\mu + b)$. As before these verifications are routine although extra care is needed in the cases of associated eigenfunctions for eigenvalues of multiplicity 2.

Table 1
Cases handled by Theorem 7

Case	Spectrum on B_k	λ_j	α_1	μ
$\mathrm{NQ}^+(1,1,1)_k$	$\lambda_{k-1} < \lambda_k < \lambda_{k+1}$	λ_k	increase	$\mu > \lambda_k$
$\mathrm{DQ}^+(1,1,1)_k$	$\lambda_{k-1} < \lambda_k < \lambda_{k+1}$	λ_k	increase	$\mu > \lambda_k$
$\mathrm{NQ}^-(1,1,1)_k$	$\lambda_k < \lambda_{k+1} < \lambda_{k+2}$	λ_{k+1}	increase	$\mu > \lambda_{k+1}$
$\mathrm{DQ}^-(1,1,1)_k$	$\lambda_k < \lambda_{k+1} < \lambda_{k+2}$	λ_{k+1}	increase	$\mu > \lambda_{k+1}$
$\mathrm{NQ}^+(2,1)_k$	$\lambda_{k-1} = \lambda_k < \lambda_{k+1}$	λ_{k-1}	increase	$\mu > \lambda_{k-1}$
$\mathrm{DQ}^+(2,1)_k$	$\lambda_{k-1} = \lambda_k < \lambda_{k+1}$	λ_{k-1}	increase	$\mu > \lambda_{k-1}$
$\mathrm{NQ}^-(2,1)_k$	$\lambda_k = \lambda_{k+1} < \lambda_{k+2}$	λ_k	increase	$\mu > \lambda_k$
$\mathrm{DQ}^-(2,1)_k$	$\lambda_k = \lambda_{k+1} < \lambda_{k+2}$	λ_k	increase	$\mu > \lambda_k$
$\mathrm{NQ}^+(1,2)_k$	$\lambda_{k-1} < \lambda_k = \lambda_{k+1}$	λ_k	decrease	$\mu < \lambda_k$
$\mathrm{NQ}^-(1,2)_k$	$\lambda_k < \lambda_{k+1} = \lambda_{k+2}$	λ_{k+1}	decrease	$\mu < \lambda_{k+1}$

Theorem 7. *For the cases listed in Table* 1, *the Crum–Darboux transformation* (7) *leads to a Sturm–Liouville problem of type* NA^+ *if* $\alpha \neq 0$, *and of type* DA^+ *if* $\alpha = 0$, *with potential*

$$Q = q - 2\left(\frac{W'(f_1, f_2)}{W(f_1, f_2)}\right)',$$

with boundary conditions (8), (9) *and with spectrum* $\{\lambda_n\colon\ n \geqslant 0,\ n \neq j\}$.

Proof. We note first that each of the cases involves three intersections (counted according to multiplicity) between $R(\lambda)$ and the kth branch of the Prüfer curve $\cot\theta(\lambda, 1)$. Since the Prüfer curve is strictly decreasing, it follows that when $a > 0$ (respectively $a < 0$) that at most one of these intersections can occur after the vertex of R (respectively, before the vertex of R). Thus at $\lambda = \lambda_j$ and at the nearby value μ, the slope of R is negative. Now we consider that $\gamma = -\frac{1}{2}(\dot{R}(\lambda_j) + \dot{R}(\mu))$ where $\dot{R}(\lambda) = \frac{d}{d\lambda}R(\lambda)$. Thus $\gamma > 0$ and the transformed problem is of the type claimed in the statement. (Note that problems of type A^+ have real simple spectrum—see, e.g., [7].) Finally, the asymptotics for λ_n given in Theorem 1, when compared with the asymptotics for problems of type A^+ (see [2,7]), show that the spectrum of the transformed problem is as stated. □

Corollary 8. *The Sturm–Liouville problems listed in Table* 1 *can be linked with two Crum–Darboux transformations to classical problems with constant boundary conditions.*

Proof. The use of a Crum–Darboux transformation with one base function (either the eigenfunction with zero oscillation number or such an eigenfunction for a slightly perturbed problem) for transforming a problem of type A^+ to a classical one is discussed in detail in [5] and indeed was one of the first cases to be studied in this general area. □

The upshot is that with a transformation using two base functions followed by one with a single base function, the cases listed in the table are linked to a classical problem establishing various examples of triples of almost isospectral problems, one member of the triple having quadratic boundary conditions, another having affine boundary conditions and the third having classical conditions.

In the next set of cases to be considered, we again use two base functions, one (f_1) being an eigenfunction from (1)–(3) corresponding to $\lambda = \lambda_j$ as before, but the other (f_2) chosen as the eigenfunction corresponding to an eigenvalue $\mu = \tilde{\lambda}_j^D$ for a perturbed problem with α replaced by α_1, where α_1 is greater than α but sufficiently close to it, and in which the right-hand boundary condition is the Dirichlet requirement, $y(1) = 0$. Thus $-f_2'' + qf_2 = \mu f_2$, $(f_2'/f_2)(0) = \cot\alpha_1$, $f_2(1) = 0$. We tabulate the possibilities as before (see Table 2).

Theorem 9. *For the cases listed in Table* 2,

(i) $W(f_1, f_2)$ *does not vanish on* $[0, 1]$.
(ii) *With*

$$e_n = \frac{W(f_1, f_2, y_n)}{W(f_1, f_2)}, \quad n \geqslant 0,\ n \neq j,$$

Table 2
Cases handled by Theorem 9

Case	Spectrum on B_k	λ_j	α_1	μ
$DQ^+(2)_0$	$\lambda_0 = \lambda_1$	λ_0	increase	$\tilde{\lambda}_0^D$
$DQ^+(1,2)_k$	$\lambda_{k-1} < \lambda_k = \lambda_{k+1}$	λ_k	increase	$\tilde{\lambda}_k^D$
$DQ^-(1,2)_k$	$\lambda_k < \lambda_{k+1} = \lambda_{k+2}$	λ_{k+1}	increase	$\tilde{\lambda}_k^D$
$NQ^+(3)_k$	$\lambda_{k-1} = \lambda_k = \lambda_{k+1}$	λ_{k-1}	increase	$\tilde{\lambda}_k^D$
$DQ^+(3)_k$	$\lambda_{k-1} = \lambda_k = \lambda_{k+1}$	λ_{k-1}	increase	$\tilde{\lambda}_k^D$
$NQ^-(3)_k$	$\lambda_k = \lambda_{k+1} = \lambda_{k+2}$	λ_k	increase	$\tilde{\lambda}_k^D$
$DQ^-(3)_k$	$\lambda_k = \lambda_{k+1} = \lambda_{k+2}$	λ_k	increase	$\tilde{\lambda}_k^D$

we have that the e_n are the eigenfunctions (and where appropriate associated eigenfunctions) corresponding to λ_n for a Sturm–Liouville problem with potential

$$Q = q - 2\left(\frac{W'(f_1, f_2)}{W(f_1, f_2)}\right)'$$

and boundary conditions (8) *at $x = 0$ and*

$$\frac{y'}{y}(1) = a\lambda^2 + B\lambda + C,$$

where B and C are constants calculable in terms of a, b, c, λ_j and μ.

(iii) *The function*

$$e_\mu = \frac{f_1}{W(f_1, f_2)}$$

is also an eigenfunction for the transformed problem for eigenvalue μ.

Proof. (i) Note that f_1 and f_2 both have k zeros in $(0, 1)$ so the result follows from Theorem 6.

(ii) The calculation of the boundary condition at $x = 0$ is as before. For the condition at $x = 1$, we use Theorem 3(i), noting that $W'(f_1, f_2)(1) = 0$. Evaluation of the expression $(W'(f_1, f_2, y_n)/W(f_1, f_2, y_n))(1)$ is tedious but routine.

(iii) The result follows from Theorem 3(iii) and direct verification of the boundary conditions. □

The effect of the transformation in these cases is to generate a new Sturm–Liouville problem with a quadratic boundary condition at $x = 1$ but with the multiplicity of the multiple eigenvalue λ_j reduced by 1 and with a new eigenvalue, μ, being introduced. Nonetheless, this new problem is included in those considered in Table 1, hence we may apply the methods there to find firstly a Crum–Darboux transformation with two base functions to reduce the problem to an almost isospectral one of type A^+ and then a subsequent transformation with a single base function to come to another almost isospectral problem with classical boundary conditions. The net result is that we have, in each of these situations, produced four almost isospectral problems: two with quadratic conditions, one with an affine condition and one with classical conditions. At each step the potential and boundary conditions can be given in terms of the original problem.

6. Transformations with complex eigenvalues

The remaining cases for study are those in which complex eigenvalues are present. For these situations, we shall list the spectrum of (1)–(3) as $\lambda_0 = \rho + i\sigma$, $(\sigma > 0)$, $\lambda_1 = \bar{\lambda}_0$, $\lambda_2 < \lambda_3 < \cdots$ with y_n being the corresponding eigenfunctions as before. Note that $y_1 = \bar{y}_0$ and that y_0 has no zeros on $(0, 1]$—see, e.g., [1]. We commence with the cases in which $\alpha \neq 0$, i.e. $\mathrm{NQ}^{\pm}\mathbb{C}$. A key result from [1] for this section is

Lemma 10. *Let $\alpha \geqslant 0$ and suppose $\lambda = \rho + i\sigma$, $\sigma > 0$, is a non-real eigenvalue for* (1)–(3) *with eigenfunction $f(x) = r(x)e^{i\Theta(x)}$ where $\Theta(0) = 0$ and $r(0) = 1$ (or $r'(0) = 1$ if $\alpha = 0$). There is $0 < \hat{\beta} < \pi$ such that for each μ sufficiently large and negative and for each $\beta_1 \in (\hat{\beta}, \pi)$ there are $\beta_0 \in (\alpha, \pi)$ and z having no zeros in $[0, 1]$ satisfying*

$$-z'' + qz = \mu z, \qquad \frac{z'}{z}(0) = \cot\beta_0, \qquad \frac{z'}{z}(1) = \cot\beta_1$$

and for which

$$\frac{r'}{r}(x) > \frac{z'}{z}(x) \quad \forall x \in [0, 1] \ \big(\forall x \in (0, 1] \text{ if } \alpha = 0\big).$$

As in [1] for the cases $\mathrm{NQ}^{\pm}\mathbb{C}$, we construct z as in Lemma 10 and use $f_1 = z$, $f_2 = y_0$, $f_3 = y_1 = \bar{y}_0$ as three base functions for a Crum–Darboux transformation so that we define

$$e_n = \frac{W(f_1, f_2, f_3, y_n)}{W(f_1, f_2, f_3)}, \quad n \geqslant 2. \tag{10}$$

The analysis of this transformation follows the corresponding theory for the affine case (when $R(\lambda) = a\lambda + b$, $a < 0$) with certain exceptions and modifications which we detail below. However we can readily claim that the e_n satisfy a Sturm–Liouville equation with potential

$$Q = q - 2\left(\frac{W'(f_1, f_2, f_3)}{W(f_1, f_2, f_3)}\right)' \tag{11}$$

and initial condition $y(0) = 0$. For the boundary condition at $x = 1$, we have, by Theorem 3(i), that

$$\frac{e_n{}'}{e_n}(1) = \frac{W'(f_1, f_2, f_3, y_n)}{W(f_1, f_2, f_3, y_n)}(1) - \frac{W'(f_1, f_2, f_3)}{W(f_1, f_2, f_3)}(1).$$

The second term of the right-hand side of this expression is constant in λ, while for the first we see that the numerator is quadratic in λ with roots at $\lambda = \lambda_0, \bar{\lambda}_0$. The denominator is cubic in λ and also has roots at $\lambda_0, \bar{\lambda}_0$. This enables us to write the boundary condition in the form

$$\frac{e_n{}'}{e_n}(1) = A - \frac{B}{\lambda + C}$$

where A, B, C are constants. In fact, we can calculate explicitly the values of these quantities, but the one of interest is

$$B = \frac{1}{a} - \frac{|\mu - \lambda_0|^2}{a|\lambda_0|^2 - c - \mu(2a\rho + b) + \cot\beta_1}. \tag{12}$$

Lemma 11. *We have* $(2a\rho + b) < 0$.

Proof. Suppose the eigenfunction corresponding to λ_0 is written as $F(x) + iG(x)$ where F and G are real functions. Note that $G(0) = G'(0) = 0$. Then separating real and imaginary parts of

$$-(F + iG)'' + q(F + iG) = (\rho + i\sigma)(F + iG),$$

we have

$$-F'' + qF = \rho F - \sigma G, \qquad -G'' + qG = \sigma F + \rho G.$$

Hence

$$FG'' - GF'' = -\sigma\left(F^2 + G^2\right), \qquad W(F,G)(1) = -\sigma \int_0^1 \left(F^2 + G^2\right).$$

However at $x = 1$ we also have

$$\frac{F' + iG'}{F + iG}(1) = a\lambda_0^2 + b\lambda_0 + c,$$
$$\frac{FF' + GG' + i(FG' - F'G)}{F^2 + G^2}(1) = \left(a\left(\rho^2 - \sigma^2\right) + b\rho + c + i\sigma(2a\rho + b)\right).$$

From the imaginary parts of this equation we obtain

$$\frac{W(F,G)(1)}{F^2(1) + G^2(1)} = \sigma(2a\rho + b)$$

and hence

$$\sigma(2a\rho + b) = \frac{-\sigma \int_0^1 (F^2 + G^2)}{F^2(1) + G^2(1)}.$$

Thus $2a\rho + b < 0$, i.e., $R'(\rho) < 0$, establishing the result. □

The consequence of this result is that the constant B in (12) will be positive provided μ is chosen sufficiently large and negative. This gives us the following result.

Theorem 12. *For the cases* $\mathrm{NQ}^{\pm}\mathbb{C}$ *the Crum–Darboux transformation* (10) *with three base functions generates a Sturm–Liouville problem with potential Q as in* (11)*, boundary conditions*

$$y(0) = 0, \qquad \frac{y'}{y}(1) = A - \frac{B}{\lambda + C} \quad \text{with } B > 0,$$

and with spectrum, $\mu, \lambda_n, n \geqslant 2$.

Proof. The only point not covered above is the fact that μ is an eigenvalue. This is easily checked;

$$e_\mu = \frac{W(f_2, f_3)}{W(f_1, f_2, f_3)}$$

is the corresponding eigenfunction and has no zeros in $(0, 1]$. Asymptotics again verify that the full spectrum of the new problem is as claimed. □

This transformed problem has a bilinear condition at $x = 1$ and is of the type discussed in [4–6], and we can now apply a further two Crum–Darboux transformations, each with a single base function, to generate an almost isospectral problem with constant boundary conditions.

The remaining cases, $\mathrm{DQ}^{\pm}\mathbb{C}$ are the least tractable mathematically and parallel the corresponding cases for the situations when $R(\lambda) = a\lambda + b$, $a < 0$—see [1]. Here we select μ large and negative and β_1 and construct z in accord with Lemma 10. Then with $\nu < \mu$ and $\tilde{\beta}_1 \geqslant \beta_1$ we repeat the construction to obtain another non-vanishing function w with

$$-w'' + qw = \nu w, \qquad \frac{w'}{w}(0) = \cot\tilde{\beta}_0, \quad \tilde{\beta}_0 > \beta_0, \qquad \frac{w'}{w}(1) = \cot\tilde{\beta}_1,$$
$$\frac{r'}{r}(x) > \frac{z'}{z}(x) > \frac{w'}{w}(x) \quad \text{on } (0, 1].$$

As in [1], it follows that $W(w, z, y_0, y_1)$ does not vanish on $[0, 1]$ and as such is suitable for use in a Crum–Darboux transformation. This leads to a new problem with potential

$$Q = q - 2\left(\frac{W'(w, z, y_0, y_1)}{W(w, z, y_0, y_1)}\right)',$$

boundary conditions

$$y(0) = 0, \qquad \frac{y'}{y}(1) = \frac{W'(w, z, y_0, y_1, y_n)}{W(w, z, y_0, y_1, y_n)} - \frac{W'(w, z, y_0, y_1)}{W(w, z, y_0, y_1)}, \tag{13}$$

spectrum $\nu < \mu < \lambda_2 < \lambda_3 < \cdots$ and corresponding eigenfunctions

$$e_v = \frac{W(z, y_0, y_1)}{W(w, z, y_0, y_1)}, \qquad e_\mu = \frac{W(w, y_0, y_1)}{W(w, z, y_0, y_1)},$$
$$e_n = \frac{W(w, z, y_0, y_1, y_n)}{W(w, z, y_0, y_1)}, \quad n \geqslant 2.$$

A comparison of asymptotics shows this listing to be the full set of eigenvalues and eigenfunctions for the transformed problem.

In the boundary condition (13), the second term on the right is constant in terms of λ ($= \lambda_n$ here), while the first has a numerator which is a fourth-order polynomial in λ with roots at λ_0, $\bar{\lambda}_0$. The denominator is cubic with roots also at λ_0, $\bar{\lambda}_0$. Thus we can rewrite the condition as

$$\frac{y'}{y}(1) = A\lambda + B + \frac{C}{\lambda + D} \tag{14}$$

for some constants A, B, C, D which depend on the choices of μ, ν, β_1 and $\tilde{\beta}_1$. While it is possible to give expressions for these constants, their form is unwieldy. Careful analysis shows that $A < 0$ for appropriate choices of μ, ν, β_1 and $\tilde{\beta}_1$, but little other information on the nature of the right-hand side of (14) is readily available. Nonetheless, the important fact is that this new problem has real simple spectrum. Further, both e_υ and e_μ have no zeros in $(0, 1]$. We can then invoke a standard (one base function) Crum–Darboux transformation using a slightly perturbed problem with potential Q, an initial condition involving $\alpha_1 > 0$ (but small) and a function h with no zeros in $[0, 1]$ satisfying the perturbed problem with eigenvalue $\xi < \nu$. This reasoning follows the arguments in [1,6] or indeed in the earlier part of this article. The outcome of this transformation is to give another problem with a non-Dirichlet initial condition and a terminal condition which we calculate as

$$\frac{y'}{y}(1) = \frac{\xi - \lambda}{A\lambda + B + \frac{C}{\lambda + D} - \left(A\xi + B + \frac{C}{\xi + D}\right)} = \frac{K\lambda + L}{M\lambda + N}$$

for new constants K, L, M, N. Thus the problem is now of a bilinear nature and is isospectral with that obtained after the first transformation. Again, and importantly, the spectrum is real and simple; we may continue the procedure with two more (single base function) Crum–Darboux transformations to next produce a problem with an affine condition, and finally one with a λ-independent condition at $x = 1$. In theory, the various coefficients appearing at each stage can be calculated, but the details are prohibitive and we content ourselves with the above descriptive approach. However the outcome is that the original DQ$^\pm$ problem is linked almost isospectrally, via the use of Crum–Darboux transformations, to a problem with boundary conditions of type (14), thence to a bilinear problem, and thence to an affine problem and so finally to a problem with constant coefficients in the boundary conditions.

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